

Math 2010 Week 13

Thm (Implicit Function Theorem)

Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open, $F: \Omega \rightarrow \mathbb{R}^k$ be C^1 ,

Denote $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_k)$

$$F(x, y) = \begin{bmatrix} F_1(x, y) \\ \vdots \\ F_k(x, y) \end{bmatrix} = \begin{bmatrix} F_1(x_1, \dots, x_n, y_1, \dots, y_k) \\ \vdots \\ F_k(x_1, \dots, x_n, y_1, \dots, y_k) \end{bmatrix}$$

Suppose $(a, b) \in \Omega$, where $a \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, such that

$$F(a, b) = c = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

and the $k \times k$ matrix

$$\left[\frac{\partial F_i}{\partial y_j}(a, b) \right]_{1 \leq i, j \leq k} = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(a, b) & \dots & \frac{\partial F_1}{\partial y_k}(a, b) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(a, b) & \dots & \frac{\partial F_k}{\partial y_k}(a, b) \end{bmatrix} \text{ is invertible}$$

Then \exists open sets $U \subseteq \mathbb{R}^n$ containing a
 $V \subseteq \mathbb{R}^k$ containing b and C^1 function

$$\varphi: U \xrightarrow{\quad} V$$
$$x = (x_1, \dots, x_n) \mapsto y = (y_1, \dots, y_k)$$

such that

- ① $\varphi(a) = b$
- ② $F(x, \varphi(x)) = c$
- ③ $\left[\frac{\partial \varphi_j}{\partial x_\ell}(a) \right]_{\substack{1 \leq j \leq k \\ 1 \leq \ell \leq n}} = - \left[\frac{\partial F_i}{\partial y_j}(a, b) \right]^{-1} \left[\frac{\partial F_i}{\partial x_\ell}(a, b) \right]_{\substack{1 \leq i \leq k \\ 1 \leq \ell \leq n}}$

Special case A: $k=1$

$$F: \Omega \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}^1$$

$$F(x, y) = F(x_1, \dots, x_n, y)$$

i.e. one constraint $F(x_1, \dots, x_n, y) = c$

Suppose $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}$

$$F(a, b) = F(a_1, \dots, a_n, b) = c$$

IFT means

If $\frac{\partial F}{\partial y}(a_1, \dots, a_n, b) \neq 0$, then

$\exists y = y(x_1, \dots, x_n)$ near (a_1, \dots, a_n)

solving the constraint (locally)

$$\begin{cases} F(x_1, \dots, x_n, y) = c \\ y(a_1, \dots, a_n) = b \end{cases}$$

In eg2 of last week

eg2 $x^2 + y^2 + z^2 = 2$. Solve for $z = z(x, y)$?

$$(x, y, z) \longleftrightarrow (x_1, x_2, y)$$

\mathbb{R}^3 Notation

General Notation

$$g(x, y, z) = x^2 + y^2 + z^2 = 2$$

$$F(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 = c$$

$$a = (a_1, a_2) = (0, 1) \quad b = 1 \quad c = 2$$

$$\frac{\partial g}{\partial z}(0, 1, 1) = 2 \neq 0$$

$$\frac{\partial F}{\partial y}(a_1, a_2, b) = 2 \neq 0$$

By IFT,

$\exists z = z(x, y)$ near $(0, 1, 1)$

such that

$$\begin{cases} g(x, y, z(x, y)) = 2 \\ z(0, 1) = 1 \end{cases}$$

By IFT,

$\exists y = y(x_1, x_2)$ near (a_1, a_2, b)

such that

$$\begin{cases} F(x_1, x_2, y(x_1, x_2)) = c \\ y(a_1, a_2) = 1 \end{cases}$$

Rmk $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ can be computed by implicit differentiation

Special case B : $n=1, k=2$

$F: \mathcal{S} \subseteq \mathbb{R}^{1+2} \rightarrow \mathbb{R}^2$ Two constraints

$$F(x, y_1, y_2) = \begin{bmatrix} F_1(x, y_1, y_2) \\ F_2(x, y_1, y_2) \end{bmatrix} \downarrow = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Suppose (a, b_1, b_2) satisfies the constraints.

then IFT means

If $\begin{bmatrix} \frac{\partial F_1}{\partial y_1}(a, b_1, b_2) & \frac{\partial F_1}{\partial y_2}(a, b_1, b_2) \\ \frac{\partial F_2}{\partial y_1}(a, b_1, b_2) & \frac{\partial F_2}{\partial y_2}(a, b_1, b_2) \end{bmatrix}$ is invertible

then $\exists y_1 = y_1(x), y_2 = y_2(x)$ near a

solving the constraints (locally)

$$\begin{cases} F_i(x, y_1(x), y_2(x)) = C_i \\ y_i(a) = b_i \end{cases}$$

for $i=1, 2$

In ex3 of last week

ex3 $\begin{cases} x^2 + y^2 + z^2 = 2 \\ x + z = 1 \end{cases}$ Solve for $y = y(x), z = z(x)$?

$$(x, y, z) \longleftrightarrow (x_1, y_1, y_2)$$

\mathbb{R}^3 Notation

General Notation

$$F_1(x, y, z) = x^2 + y^2 + z^2 = 2$$

$$F_2(x, y, z) = x + z = 1$$

$$F_1(x, y_1, y_2) = x^2 + y_1^2 + y_2^2 = C_1$$

$$F_2(x, y_1, y_2) = x + y_2 = C_2$$

$$a=0, b=(b_1, b_2)=(1, 1), c=(c_1, c_2)=(2, 1)$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}_{(0,1,1)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

invertible

By IFT,

$\exists y = y(x), z = z(x)$ near $(0, 1, 1)$ such that

$$\begin{cases} F_1(x, y(x), z(x)) = 2 \\ F_2(x, y(x), z(x)) = 1 \\ y(0) = 1 \\ z(0) = 1 \end{cases}$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}_{(a,b_1,b_2)} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

invertible

By IFT,

$\exists y_1 = y_1(x), y_2 = y_2(x)$ near (a, b_1, b_2) such that

$$\begin{cases} F_i(x, y_1(x), y_2(x)) = C_i \\ y_i(a) = b_i \end{cases}$$

for $i=1, 2$

Rmk We write $F = F(x, y)$ and solve y as a function of x , but in fact, the ordering of the variables is not important.

eg Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8 \\ x + 4y + 3z = 18 \end{cases}$$

Near $(2, 1, 4)$, can we solve 2 of the variables as functions of the remaining variable?

Sol

$$\begin{aligned} \text{Let } F(x, y, z) &= \begin{bmatrix} F_1(x, y, z) \\ F_2(x, y, z) \end{bmatrix} \\ &= \begin{bmatrix} xz + \sin(yz - x^2) \\ x + 4y + 3z \end{bmatrix} \end{aligned}$$

$$\nabla F = \begin{bmatrix} z - 2x \cos(yz - x^2) & z \cos(yz - x^2) & x + y \cos(yz - x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$$\nabla F(2, 1, 4) = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 0 & 4 \\ 1 & 4 \end{vmatrix} = -4 \neq 0 \xrightarrow{\text{IFT}} \begin{array}{l} x, y \text{ can be expressed as} \\ \text{functions of } z \text{ near } (2, 1, 4) \end{array}$$

$$\begin{vmatrix} 0 & 3 \\ 1 & 3 \end{vmatrix} = -3 \neq 0 \xrightarrow{\text{IFT}} \begin{array}{l} x, z \text{ can be expressed as} \\ \text{functions of } y \text{ near } (2, 1, 4) \end{array}$$

$$\begin{vmatrix} 4 & 3 \\ 4 & 3 \end{vmatrix} = 0 \quad \begin{array}{l} \text{No Conclusion on whether } y, z \text{ can be} \\ \text{expressed as functions of } x \text{ near } (2, 1, 4) \\ \text{from IFT} \end{array}$$

Rmk y, z are not differentiable of x near $(2, 1, 4)$.

Otherwise, by implicit differentiation,

$$\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{no solution} \Rightarrow \text{contradiction}$$

Rmk Implicit Function Theorem has many important applications, such as rigorous proofs of

- ① Implicit differentiation
- ② Tangent plane of surface $F(x, y, z) = C$
- ③ Lagrange Multipliers

The following Inverse Function Theorem is equivalent to Implicit Function Theorem

Thm (Inverse Function Theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be open,

$f: \Omega \rightarrow \mathbb{R}^n$ be C^1 , $f(a) = b$

Suppose $Df(a)$ is an invertible $n \times n$ matrix.

Then \exists open sets $U_1, U_2 \subseteq \mathbb{R}^n$, $a \in U_1$, $b \in U_2$

and a C^1 function $g: U_2 \rightarrow U_1$ such that

① $g(b) = a$

② $g(f(x)) = x \quad \forall x \in U_1$

$f(g(y)) = y \quad \forall y \in U_2$

(g is a local inverse of $f : g = (f|_{U_1})^{-1}$)

③ $Dg(b) = Df(a)^{-1}$

eg $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x,y) = (x^2 - y^2, 2xy)$$

Clearly $f(-x, -y) = f(x, y)$

$\Rightarrow f$ is not injective and has no global inverse

How about local inverse?

Soln $f(x,y) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$

$$Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$\det Df(x,y) = 4(x^2 + y^2) > 0 \Leftrightarrow (x,y) \neq (0,0)$$

By Inverse Function Theorem f is locally invertible with differentiable local inverse

For instance, let $g(u,v)$ be a local inverse of $f(x,y)$ near $(x,y) = (1,-1)$

Then $f(1,-1) = (0,-2) \Rightarrow g(0,-2) = (1,-1)$

$$Dg(0,-2) = Df(1,-1)^{-1} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

In fact, we can find $g(u,v)$ explicitly.

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$$

Near $(x,y) = (1,-1)$, $x \neq 0 \Rightarrow y = \frac{v}{2x}$

$$\therefore u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$4x^4 - 4ux^2 - v^2 = 0$$

$$\begin{aligned} x^2 &= \frac{4u \pm \sqrt{(-4u)^2 - 4(4)(-v^2)}}{8} \\ &= \frac{u \pm \sqrt{u^2 + v^2}}{2} \end{aligned}$$

Put $(x, y) = (1, -1)$, $(u, v) = (0, -2)$

$$\Rightarrow r^2 = \frac{0 \pm \sqrt{4}}{2}$$

$$\Rightarrow \text{"-"} \text{ is rejected, } x^2 = \frac{u + \sqrt{u^2 + v^2}}{2}$$

$$\Rightarrow x = \sqrt{\frac{u + \sqrt{u^2 + v^2}}{2}}$$

$$y = \frac{2v}{x} = \frac{2\sqrt{2}v}{\sqrt{u + \sqrt{u^2 + v^2}}}$$

Hence,

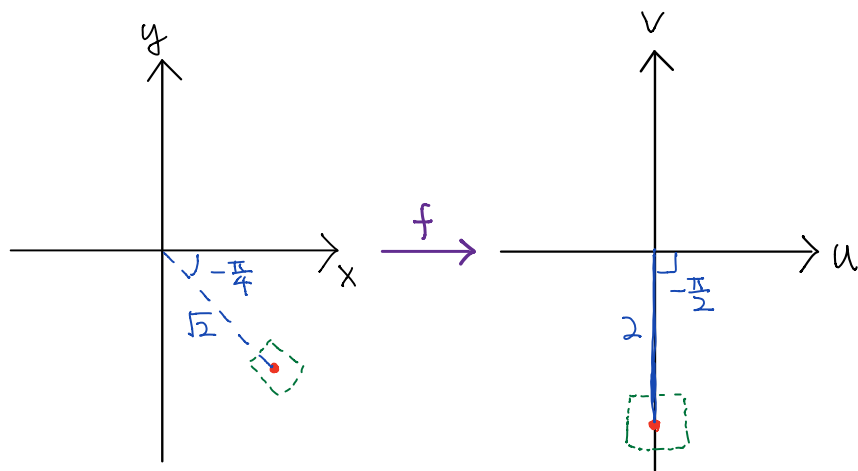
$$g(u, v) = \left(\sqrt{\frac{u + \sqrt{u^2 + v^2}}{2}}, \frac{2\sqrt{2}v}{\sqrt{u + \sqrt{u^2 + v^2}}} \right)$$

Rmk If $z = x + yi \in \mathbb{C}$, then

$$z^2 = x^2 - y^2 + 2xyi$$

$$\therefore u = x^2 - y^2 = \operatorname{Re} z^2$$

$$v = 2xy = \operatorname{Im} z^2$$



$$z \longmapsto z^2$$

In both Implicit and Inverse Function Theorems,

we assume a Jacobi matrix to be invertible.

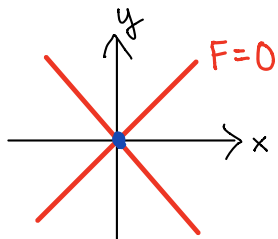
Without this assumption, the theorems are inconclusive on the

existence of local implicit or inverse function. See examples below:

Implicit Function Theorem

$$F(x,y) = x^2 - y^2 = 0$$

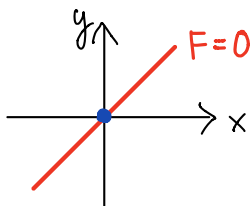
$$\left. \frac{\partial F}{\partial y} \right|_{(0,0)} = 0$$



y is not locally a function of x near $(0,0)$

$$F(x,y) = x^3 - y^3 = 0$$

$$\left. \frac{\partial F}{\partial y} \right|_{(0,0)} = 0$$

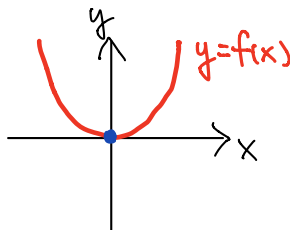


$y = x$ both globally and locally near $(0,0)$

Inverse Function Theorem

$$f(x) = x^2$$

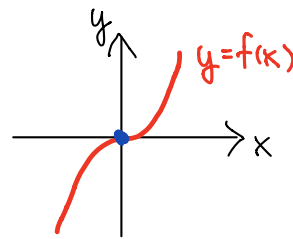
$$f'(0) = 0$$



Not injective near $x=0$
 \Rightarrow no local inverse near $x=0$

$$f(x) = x^3$$

$$f'(0) = 0$$



$g(y) = \sqrt[3]{y}$ is both global and local inverse near $x=0$